Journal of Statistical Physics, Vol. 115, Nos. 3/4, May 2004 (© 2004)

Asymptotic Behavior of the Integrated Density of States of Acoustic Operators with Random Long Range Perturbations

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Received August 13, 2003; accepted December 3, 2003

In this paper we study the behavior of the integrated density of states of random acoustic operators of the form $A_\omega = -\nabla \frac{1}{\varrho_\omega} \nabla$. When ϱ_ω is considered as an Anderson type long range perturbations of some periodic function, the behavior of the integrated density of states of A_ω in the vicinity of the internal spectral edges is given.

KEY WORDS: Spectral theory; random operators; integrated density of states; Lifshitz tails.

1. INTRODUCTION

Basic properties of wave propagation in a nonhomogeneous medium eventually boil down to the spectral properties of the relevant self-adjoint differential operator. As far as the acoustic waves are concerned, they are governed by the so called "acoustic operator."[(2)] It is a self adjoint operator on $L^2(\mathbb{R}^d)$ and formally defined by:

$$A_\omega = A(\varrho_\omega) = -\nabla \frac{1}{\varrho_\omega} \nabla = -\sum_{i=1}^{d} \partial_{x_i} \frac{1}{\varrho_\omega} \partial_{x_i}, \tag{1.1}$$

where ϱ_ω is a positive and bounded function which represents the mass density or the elasticity contrast of the medium where the wave propagates. See ref. 2 and references therein for more physicals interpretations and motivation.

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Let us start by defining the main object of our study, the integrated density of states. For this we consider Λ a cube of $\mathbb{R}^d$. We note by $A_{\omega,\Lambda}$ the restriction of A_ω to Λ with self adjoint boundary conditions. As A_ω is elliptic, the resolvent of $A_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $A_{\omega,\Lambda}$ is discrete and is made of isolated eigenvalues of finite multiplicity.[22] We define

$$N_\Lambda(E) = \frac{1}{\mathrm{vol}(\Lambda)} \cdot \#\{\text{eigenvalues of } A_{\omega,\Lambda} \leqslant E\}. \tag{1.2}$$

Here vol(Λ) is the volume of Λ in the Lebesgue sense and $\#E$ is the cardinal of E.

It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^d$ exists almost surely and is independent of the boundary conditions. It is called the *integrated density of states* of A_ω (IDS for the short form). See ref. 21.

The study of the integrated density of states and specially of its behavior is of interest for its relationship with physical interpretations. See ref. 5. The question we are interested in here deals with the behavior of N at the internal spectral edges of A_ω.

1.1. The Behavior of the IDS

We start by giving a brief history of the subject. In 1964 and under physical considerations Lifshitz[14] argued that, for a Schrödinger operator of the form $H_\omega = -\Delta + V_\omega$, there exists $c_1, c_2 > 0$ such that $N(E)$ satisfies the asymptotic:

$$N(E) \simeq c_1 \exp(-c_2(E-E_0)^{-\alpha}), \qquad E \to E_0. \tag{1.3}$$

Here E_0 is the bottom of the spectrum of H_ω and $\alpha > 0$. The behavior (1.3) is known as *Lifshitz tails* (for more details see part IV.9.A of ref. 21). Lifshitz predicted (1.3) also at fluctuating edges inside the spectrum. The latter are those parts of the spectrum which are determined by rather rare events.

The principal results known on Lifshitz tails are mainly shown for Schrödinger operators (for continuous and discrete cases). (See refs. 5, 7, 9, 20, 23, and others.) For an operator of type (1.1), see refs. 15, 16, and 18.

1.2. The Result

When ϱ_ω in (1) is obtained by a short range perturbation of Anderson type of some periodic function, it is proved[16] that the IDS of A_ω exhibits internal Lifshitz tails at the edges of the spectral gaps if and only if the IDS of some periodic operator is non degenerate at the same edges. The essential goal of this work is to give the asymptotic of the IDS of the operator defined by (1) when ϱ_ω is given as a long range perturbation of some periodic function. Note that the main novelty of this paper compared to ref. 16 is that the asymptotic is given without any assumption on the behavior of the IDS of the background operator. As we will see, in contrast to ref. 16, in the present situation the kinetic energy, i.e., the Floquet eigenvalues of the background periodic operator do not influence the IDS behavior (See also ref. 18 where we require to the dimension to be 2.). Indeed in this case the Lifshitz exponent (the α in Eq. (1.3)) does not depend on the uncertainly principle, i.e., on the kinetic energy. We refer to this situation as the classical regime. Note that here as in ref. 16 we consider the case where the decreasing rate of the probability density at the edges of its support is 0. See (H.3).

The proof of the result is based on the use of the technique of periodic approximations[9, 16] and is composed of two main parts, the upper and the lower bounds.

To present our result we consider the following plan:

In Section 2, we define the model to be studied and specify various assumptions. We introduce a periodic reference operator A_{ω^+}. We state the principal theorem (Theorem 2.1) which gives the asymptotic of the IDS.

To prove Theorem 2.1, the technique of the periodic approximations[9, 16] enables to approximate the IDS of A_ω with that of well chosen periodic operators. This will be done in Sections 3. Section 4 is devoted to the proof of Theorem 2.1.

2. THE MODEL

Let us start this section by giving the expression of ϱ_ω. We assume that ϱ_ω is a function which satisfies

(H.0)

$$\varrho_\omega = \varrho_0 \left(1 + \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma u_\gamma \right),$$

where

(i) ϱ_0 is measurable real and $\mathbb{Z}^d$-periodic function, i.e.,

$$\varrho_0(x) = \varrho_0(x+\gamma), \qquad \forall x \in \mathbb{R}^d, \quad \gamma \in \mathbb{Z}^d.$$

(ii) There exists constants $\varrho_{0,+} > \varrho_{0,-} > 0$ such that for almost all $x \in \mathbb{R}^d$, we have

$$0 < \varrho_{0,-} \leqslant \varrho_0(x) \leqslant \varrho_{0,+}. \tag{2.4}$$

(iii) For $\gamma \in \mathbb{Z}^d$, we set $u_\gamma(\cdot) = u(\cdot - \gamma)$. We suppose that u is a real function such that there exists $U_+ > 0$: such that for almost all $x \in \mathbb{R}^d$

$$0 \leqslant U(x) \equiv \sum_{\gamma \in \mathbb{Z}^d} u_\gamma(x) \leqslant U_+ < \infty. \tag{2.5}$$

(iv) $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d}$ is a family of non constant and positive, independent identically distributed random variables whose common probability measure is denoted by $\mathbb{P}_{\omega_0}$. We denote the probability space by $(\Omega, \mathscr{F}, \mathbb{P})$. We assume that $\mathbb{P}_{\omega_0}$ is compactly supported.

Let $\mathscr{A}(\varrho_\omega)$ be the quadratic form defined as follow: for $u \in H^1(\mathbb{R}^d) = \mathscr{D}(\mathscr{A}(\varrho_\omega))$

$$\mathscr{A}(\varrho_\omega)[u, u] = \int_{\mathbb{R}^d} \frac{1}{\varrho_\omega(x)} \nabla u(x) \, \overline{\nabla u(x)} \, dx.$$

$\mathscr{A}(\varrho_\omega)$ is a symmetrical, closed and positive quadratic form. A_ω given by (1) is defined to be the self adjoint operator associated to $\mathscr{A}(\varrho_\omega)$.[22]

Assumption (H.0) ensures that A_ω is a measurable family of self adjoint operators and ergodic.[5, 21] Indeed, if τ_γ refers to the translation by γ, then $(\tau_\gamma)_{\gamma \in \mathbb{Z}^d}$ is a group of unitary operators on $L^2(\mathbb{R}^d)$ and for $\gamma \in \mathbb{Z}^d$ we have

$$\tau_\gamma A_\omega \tau_{-\gamma} = A_{\tau_\gamma \omega}.$$

According to refs. 5, 21 we know that there exists Σ, Σ_{pp}, Σ_{ac}, and Σ_{sc} closed and non random sets of $\mathbb{R}$ such that Σ is the spectrum of A_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of A_ω, then $\Sigma_{pp} = \sigma_{pp}$, $\Sigma_{ac} = \sigma_{ac}$, and $\Sigma_{sc} = \sigma_{sc}$ with probability one.

2.1. Reference Operator

It is convenient to consider A_ω as a perturbation of some periodic operator A_{ω^+}. More precisely, for $\varrho_{\omega^+} = \varrho_0(1 + \omega^+ \sum_{\gamma \in \mathbb{Z}^d} u_\gamma)$, where $\omega^+ = \sup(\operatorname{supp} \mathbb{P}_{\omega_0})$ we write:

$$A_\omega = A_{\omega^+} + \Delta A_\omega,$$

with

$$A_{\omega^+} = A(\varrho_{\omega^+})$$

and

$$\Delta A_\omega = A_\omega - A_{\omega^+} = -\nabla \frac{\varrho_{\omega^+} - \varrho_\omega}{\varrho_{\omega^+}\varrho_\omega} \nabla \geqslant 0.$$

2.1.1. Main Assumptions

We assume that

(H.1)

- There exists E_+ and $\delta > 0$ such that $\sigma(A_{\omega^+}) \cap [E_+, E_+ + \delta) = [E_+, E_+ + \delta)$ and $\sigma(A_{\omega^+}) \cap (E_+ - \delta, E_+] = \emptyset$.

To prove our result, we will need the following assumptions:

(H.2)

- Let $C_0 = \{x \in \mathbb{R}^d; \forall 1 \leqslant j \leqslant d; -\frac{1}{2} < x_j \leqslant \frac{1}{2}\}$. There exists $\nu \in (d, d+2]$ and $0 \leqslant g_- \leqslant g_+$ two non vanishing functions on $L^2(C_0)$, such that for any $\gamma \in \mathbb{Z}^d$ and almost every $x \in C_0$ one has

$$g_-(x) \leqslant u(x+\gamma) \cdot (1+|\gamma|)^\nu \leqslant g_+(x),$$

and for all $1 \leqslant i \leqslant d$,

$$g_-(x) \leqslant |(\partial_{x_i} u)(x+\gamma)| \cdot (1+|\gamma|)^\nu \leqslant g_+(x).$$

(H.3)

- $\limsup_{\varepsilon \to 0^+} \frac{\log |\log \mathbb{P}_{\omega^0}([\omega^+ - \varepsilon, \omega^+])|}{\log \varepsilon} = 0.$

As, $\Delta A_\omega \geqslant 0$ and ω^+ is in the support of $\mathbb{P}_{\omega_0}$, Σ contains an interval of the form $[E_+, E_+ + a] (a > 0)$ (see ref. 8).

As we are interested in the behavior of the IDS in the neighborhood of E_+, we require that E_+ remains always the edge of a gap for Σ, when the perturbation is turned on. More precisely, if for all $t \in [0, 1]$, we define $A_{\omega, t} = A_{\omega^+} + t\, \Delta A_\omega$ and Σ_t is the almost sure spectrum of $A_{\omega, t}$, then one requires that the following assumption hold.

(H.4)

There exists $\delta' > 0$ such that for all $t \in [0, 1]$, $\Sigma_t \cap [E_+ - \delta', E_+) = \emptyset$.

2.1.2. The Main Theorem

The main result of this work is:

Theorem 2.1. Let A_ω be the operator defined by (1). We assume that (H.1)–(H.4) hold. Then E_+ is a continuity point for N and

$$\lim_{\varepsilon \to 0,\, \varepsilon > 0} \frac{\log |\log(N(E_+ + \varepsilon) - N(E_+))|}{\log \varepsilon} = -\frac{d}{\nu - d}. \tag{2.6}$$

Remark 2.2.

– The result of Theorem 2.1 is stated for lower band edges. Under adequate assumptions the corresponding result can be proved for upper band edges.

– As it has already been mentioned in Remark 11 of ref. 2, one can use the result of Theorem 2.1 to show either Anderson localization[2] or dynamical localization[1] under assumptions on the distribution of the random variables weaker than those required in these references. This was done in the Schrödinger case in ref. 25 and for the divergence operator in ref. 17.

Outline of the Proof. To show Theorem 2.1, we use periodic approximations. This technique allows us to approximate exponentially the initial IDS (see Lemma 3.3), by that of some periodic operators. Then we have just to control the behavior of the IDS of those periodic operators and take the limit. To do this, the upper and lower bounds are proven separately.

Note that in ref. 16 the upper bound is proved under the non degeneracy assumption of the IDS of the background operator A_{ω^+} which is relaxed here. The upper bound is proved by the use of probabilistic arguments and Markov inequalities.[11]

The lower bound is proved by constructing a large enough number of orthogonal approximate eigenfunctions of $A_{\omega,\Lambda}$ associated with approximate eigenvalues in $[E_+ - \varepsilon, E_+ + \varepsilon]$. This, will enables to lower bound the number of the eigenvalues of $A_{\omega,\Lambda}$ in the interval $[E_+ - \varepsilon, E_+ + \varepsilon]$.

We end this section by remarks about our assumptions. Let us start with (H.1). Figotin and Kuchment in ref. 3 studied the existence of open spectral gaps in the spectrum of certain periodic acoustic operators for $d = 2$ and 3. In assumption (H.1) we asked that $E_+ > 0$ which excludes the spectral gap $(-\infty, 0)$. Lifshitz tails is likely to occur at the neighborhood of the fluctuation edges. See ref. 21. It should be noted that 0 is not a fluctuation edge of the spectrum. It belongs to the spectrum of A_ω independently of the choice of $\varrho_\omega(x)$.

If the support of $\mathbb{P}_{\omega_0}$ is connected, the assumption (H.4) can be replaced by:

(H.4.bis). There exists $\delta' > 0$ such that $\Sigma \cap [E_+ - \delta', E_+) = \emptyset$.

By adding a disorder parameter g in the equation which defines ϱ_ω, i.e.,

$$\varrho_\omega = \varrho_0 \left(1 + g \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma u_\gamma \right),$$

we can choose g small enough so that the spectral gap in $\sigma(A_{\omega^+})$ will not be closed after the perturbation.[2]

3. APPROXIMATION OF THE DENSITY OF STATES

For completeness, in this section we review some of the interesting properties of the periodic operators,[13] then we will approximate the density of states of A_ω by the density of states of periodic approximations. In a neighborhood of E_+, we will control the behavior of the density of states of periodic approximations via the density of states of periodic approximations of the reference operators. We then compute the limit for the density of states of the reference operators and we obtain the sought for result.

3.1. Some Floquet Theory

Now we review some standard facts from the Floquet theory for periodic operators. Basic references for this material are.[13, 22, 24]

As ϱ_{ω^+} is $\mathbb{Z}^d$-periodic, for any $\gamma \in \mathbb{Z}^d$, we have

$$\tau_\gamma A_{\omega^+} \tau_\gamma^* = \tau_\gamma A_{\omega^+} \tau_{-\gamma} = A_{\omega^+}.$$

Let $\mathbb{T}^* = \mathbb{R}^d / (2\pi \mathbb{Z}^d)$. We define $\mathscr{H}$ by

$$\mathscr{H} = \{ u(x, \theta) \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \otimes L^2(\mathbb{T}^*); \forall (x, \theta, \gamma) \in \mathbb{R}^d \times \mathbb{T}^* \times \mathbb{Z}^d; u(x + \gamma, \theta) = e^{i\gamma\theta} u(x, \theta) \}.$$

There exists U a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathscr{H}$ such that A_{ω^+} admits the Floquet decomposition[13, 24]

$$UA_{\omega^+}U^* = \int_{\mathbb{T}^*}^{\oplus} A_{\omega^+}(\theta)\, d\theta.$$

Here $A_{\omega^+}(\theta)$ is the operator A_{ω^+} acting on $\mathscr{H}_\theta$, defined by

$$\mathscr{H}_\theta = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^d);\ \forall \gamma \in \mathbb{Z}^d,\ u(x+\gamma) = e^{i\gamma\theta} u(x)\}.$$

As A_{ω^+} is elliptic, we know that, $A_{\omega^+}(\theta)$ has a compact resolvent; hence its spectrum is discrete.[22] We denote its eigenvalues, called Floquet eigenvalues of A_{ω^+}, by

$$E_0(\theta) \leqslant E_1(\theta) \leqslant \cdots \leqslant E_n(\theta) \leqslant \cdots.$$

The corresponding eigenfunctions are denoted by $(w(x,\cdot)_j)_{j \in \mathbb{N}}$. The functions $(\theta \to E_n(\theta))_{n \in \mathbb{N}}$ are Lipshitz-continuous, and we have

$$E_n(\theta) \to +\infty \qquad \text{as} \quad n \to +\infty \quad \text{uniformly in } \theta.$$

The spectrum $\sigma(A_{\omega^+})$ of A_{ω^+} is made of bands (i.e., $\sigma(A_{\omega^+}) = \bigcup_{n \in \mathbb{N}} E_n(\mathbb{T}^*)$).

3.2. The Periodic Approximations

Let $k \in \mathbb{N}^*$. We define the following periodic operator

$$A_{\omega,k} = -\nabla \frac{1}{\varrho_{\omega,k}} \nabla,$$

where the function $\varrho_{\omega,k}$ is defined by

$$\varrho_{\omega,k} = \varrho_0 \left(1 + \sum_{\gamma \in C_k \cap \mathbb{Z}^d} \omega_\gamma \sum_{\beta \in (2k+1)\mathbb{Z}^d} u_{\gamma+\beta}\right),$$

C_k is the cube

$$C_k = \left\{x \in \mathbb{R}^d;\ \forall 1 \leqslant j \leqslant d,\ -\frac{2k+1}{2} < x_j \leqslant \frac{2k+1}{2}\right\}.$$

$A_{\omega,k}$ is $(2k+1)\,\mathbb{Z}^d$-periodic and essentially self adjoint operator. Let $\mathbb{T}_k^* = (\mathbb{R}^d)/\frac{2\pi}{2k+1}\,\mathbb{Z}^d$. We define $N_{\omega,k}$ the IDS of $A_{\omega,k}$ by

$$N_{\omega,k}(E) = \frac{1}{(2\pi)^d} \sum_{n\in\mathbb{N}} \int_{\{\theta\in\mathbb{T}_k^*,\, E_{\omega,k,n}(\theta)\leqslant E\}} d\theta. \tag{3.7}$$

Let $dN_{\omega,k}$ the derivative of $N_{\omega,k}$ in the distribution sense. As $N_{\omega,k}$ is increasing, $dN_{\omega,k}$ is a positive measure; it is the density of states of $A_{\omega,k}$. We denote by dN the density of states of A_ω. For all $\varphi\in C_0^\infty(\mathbb{R})$, $dN_{\omega,k}$ verifies (see refs. 9 and 22)

$$\begin{aligned}\langle\varphi, dN_{\omega,k}\rangle &= \frac{1}{(2\pi)^d} \int_{\theta\in\mathbb{T}_k^*} \operatorname{tr}_{\mathscr{H}_\theta}(\varphi(A_{\omega,k,\theta}))\, d\theta,\\ &= \frac{1}{\operatorname{vol}(C_k)} \operatorname{tr}(\chi_{C_k}\varphi(A_{\omega,k})\,\chi_{C_k}),\end{aligned} \tag{3.8}$$

where for $\Lambda\subset\mathbb{R}^d$, χ_Λ will design the characteristic function of Λ and $\operatorname{tr}(A)$ is the trace of A (we index by $\mathscr{H}_\theta$ if the trace is taken in $\mathscr{H}_\theta$). The proof of (3.8) is given in ref. 9.

Theorem 3.1.

(1) For any $\varphi\in C_0^\infty(\mathbb{R})$ and for almost all $\omega\in\Omega$ we have

$$\lim_{k\to\infty} \langle\varphi, dN_{\omega,k}\rangle = \langle\varphi, dN\rangle.$$

(2) For any $\lambda\in\mathbb{R}$ a continuity point for N, we have $\lim_{k\to\infty} \mathbb{E}(N_{\omega,k}(\lambda)) = N(\lambda)$ almost surely.

Remark 3.2. The result of Theorem 3.1 is close to that of Theorem 5.1 of ref. 9. The proof is also similar and is based on functional analysis. The unique difference in the proof comes from the control of the behavior of the resolvent. In ref. 9, the perturbation is a potential; in our case, it is a differential operator of the same order as the background operator. The detail of the proof is given in ref. 16.

In what follow we prove that the IDS of A_ω is exponentially well approximated by the expectation of the IDS of the periodic operators $A_{\omega,k}$ when k is polynomial in ε^{-1}. More precisely we prove

Lemma 3.3. For any $\eta_0 > 1$ and $I \subset \mathbb{R}$ a compact interval, there exists $\nu_0 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, $E \in I$ and $k \geqslant k_1 = \varepsilon^{-\nu_0}$, we have

$$\begin{aligned}\mathbb{E}[N_{\omega,k}(E+\varepsilon/2) - N_{\omega,k}(E-\varepsilon/2)] - e^{-\varepsilon^{-\eta_0}} \\ \leqslant N(E+\varepsilon) - N(E-\varepsilon) \\ \leqslant \mathbb{E}[N_{\omega,k}(E+2\varepsilon) - N_{\omega,k}(E-2\varepsilon)] + e^{-\varepsilon^{-\eta_0}}.\end{aligned} \tag{3.9}$$

Proof. The last result is well known for operators with compact single site potentials.[10–12] For this we need to define another operator. More precisely let f a function on $\mathbb{R}^d$, we set $f^\varepsilon(x) = f(x)\,\chi_{\{\varepsilon \cdot |x| \leqslant 1\}}$, f^ε is compactly supported. We define the following random operator:

$$A_{\varepsilon,\omega} = -\nabla \cdot \frac{1}{\varrho_{\varepsilon,\omega}} \cdot \nabla,$$

where $\varrho_{\varepsilon,\omega}(\cdot)$ is the function given by

$$\varrho_{\varepsilon,\omega}(\cdot) = \varrho_0 \left(1 + \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma u^\varepsilon(\cdot - \gamma)\right).$$

The periodic approximations of $A_{\varepsilon,\omega}$ it is defined analogously to $A_{\omega,k}$ and will be denoted by $A_{\varepsilon,\omega,k}$. Let N_ε (respectively $N_{\varepsilon,\omega,k}$) be the IDS of $A_{\varepsilon,\omega}$ (respectively of $A_{\varepsilon,\omega,k}$). Notice that from the decaying assumption (H.2) and the fact that the random variables are bounded, uniformly in k and ω, we get that there exits $C > 0$ such that for any $\Psi = (-1-\Delta)^{-1}\varphi$, where $\varphi \in L^2(\mathbb{R}^d)$

$$0 \leqslant \langle A_\omega \Psi, \Psi \rangle - \langle A_{\varepsilon,\omega} \Psi, \Psi \rangle \leqslant C \cdot \varepsilon^{\nu-d} \cdot \|\Psi\|^2.$$

The same inequality holds for the periodic approximations. This yields that, uniformly in k and ω and locally uniformly in the energy E we have

$$N_\varepsilon(E - C \cdot \varepsilon^{\nu-d}) \leqslant N(E) \leqslant N_\varepsilon(E + C \cdot \varepsilon^{\nu-d}), \tag{3.10}$$

and

$$N_{\varepsilon,\omega,k}(E - C \cdot \varepsilon^{\nu-d}) \leqslant N_{\omega,k}(E) \leqslant N_{\varepsilon,\omega,k}(E + C \cdot \varepsilon^{\nu-d}). \tag{3.11}$$

This tells us that the IDS of A_ω is well approximated by that of $A_{\varepsilon,\omega}$. But for the last operator the singles sites potentials are compactly supported and so many techniques and results are available.[11] We need the following lemma

Lemma 3.4 [11]. We assume that the single site potential is supported in a ball of radius R_ε. Let I, a relatively compact open interval in $\mathbb{R}$. For any $\beta \in (0,1)$, there exists $C>1$ and $\rho>0$ such that, for any $\varphi \in C_0^\infty(I)$, for $n \in \mathbb{N}^*$ and $k > R_\varepsilon$, we have

$$|\mathbb{E}(\langle \varphi, dN_{\omega,k}\rangle) - \langle \varphi, dN\rangle| \leqslant C \cdot |k-R_\varepsilon|^{-(1-\beta)k} \cdot n^n \cdot \sup_{x\in\mathbb{R},\, 0\leqslant j\leqslant n+\rho} |(|x|+1)^{\rho+n}\, \varphi^{(j)}(x)|. \tag{3.12}$$

Remark 3.5. This lemma is proven in ref. 11 for the Schrödinger case. It is still true for our case. The proof is based on the Helffer Sjöstrand formula and the resolvent equation with the exponential decay of the resolvent kernels (the Combes–Thomas argument).

Let φ be a Gevrey class function, of Gevrey exponent $\alpha>1$ (see ref. 4) such that φ is supported in $[-2,2]$, $0\leqslant\varphi\leqslant 1$ and $\varphi\equiv 1$ on $[-1,1]$. For $0<\varepsilon<1$ and $E\in\mathbb{R}$, we set

$$\varphi_{E,\varepsilon}(\cdot) = \varphi\left(\frac{\cdot - E}{\varepsilon}\right).$$

Fix $I\subset\mathbb{R}$ compact. Then from Lemma 3.4 and properties of Gevrey class functions, we deduce that there exists $C>1$ such that for $E\in I$, $k>R_\varepsilon$, $n\geqslant 1$ and $0<\varepsilon<1$ we have

$$|\mathbb{E}(\langle \varphi_{E,\varepsilon}, dN_{\omega,k}\rangle) - \langle \varphi_{E,\varepsilon}, dN\rangle| \leqslant \varepsilon^{-n-\rho}\,(n+\rho)^{2\alpha(n+\rho)}\,(k-R_\varepsilon)^{-(1-\beta)n}.$$

We take $n\leqslant (k-R_\varepsilon)^{(1-\beta)/4\alpha}-\rho$. For $k-R_\varepsilon$ large, we get that, there exists $C>1$ such that for $n>\rho$ and $0<\varepsilon<1$, we have

$$|\mathbb{E}(\langle \varphi_{E,\varepsilon}, dN_{\omega,k}\rangle) - \langle \varphi_{E,\varepsilon}, dN\rangle| \leqslant (\varepsilon^{-1}(k-R_\varepsilon)^{(1-\beta)/4})^{(k-R_\varepsilon)^{(1-\beta)/4\alpha}}.$$

As $R_\varepsilon\sim\varepsilon^{1/(d-\nu)}$, for $\eta_0>1$ such that $\alpha\cdot\eta_0>1$ and $k=k_1=\varepsilon^{-\nu_0}>\varepsilon^{-\eta_0 4\alpha/(1-\beta)}+R_\varepsilon$, (it suffice to take $\nu_0>\sup(\eta_0 4\alpha/(1-\beta), \frac{1}{d-\nu}))$ we get that there exist $\varepsilon_0>0$ such that, for $0<\varepsilon<\varepsilon_0$, we have

$$|\mathbb{E}(\langle \varphi_{E,\varepsilon}, dN_{\omega,k_\varepsilon}\rangle) - \langle \varphi_{E,\varepsilon}, dN\rangle| \leqslant e^{-\varepsilon^{-\eta_0}}. \tag{3.13}$$

As dN_ε and $dN_{\varepsilon,\omega,k}$ are positive measures and by the definition of φ, we get

$$\begin{aligned}\mathbb{E}(N_{\varepsilon,\omega,k}(E+\varepsilon)-N_{\varepsilon,\omega,k}(E-\varepsilon)) &\leqslant \mathbb{E}(\langle\varphi_{E,\varepsilon}, dN_{\varepsilon,\omega,k}\rangle)\\ &\leqslant \mathbb{E}(N_{\varepsilon,\omega,k}(E+2\varepsilon)-N_{\varepsilon,\omega,k}(E-2\varepsilon))\end{aligned}$$

and

$$N_\varepsilon(E+\varepsilon)-N_\varepsilon(E-\varepsilon)\leqslant\langle\varphi_{E,\varepsilon}, dN\rangle\leqslant N_\varepsilon(E+2\varepsilon)-N_\varepsilon(E-2\varepsilon).$$

This, and Eq. (3.13) gives (3.9), for $A_{\varepsilon,\omega}$. To get (3.9) for A_ω, it suffices to pick $\varepsilon=\varepsilon^{1/\nu-d}$ and take into account (3.10), (3.11), and (3.13). ▮

Now we have the necessary tools to prove the main theorem.

4. THE PROOF OF THEOREM 2.1

Notice that the first point of the theorem, i.e., the continuity of the IDS at E_+ is a consequence of the continuity if the IDS of the periodic operator. See ref. 16.

As we mentioned in the introduction the proof of the main result is composed of two classical parts, the upper and the lower bounds. We start by the proof of the upper bound, then we turn to the lower bound.

4.1. The Upper Bound

From Lemma 3.3 and for $\eta_0>1/(\nu-d)$ and $k\sim\varepsilon^{-\delta}$ such that $\delta>\nu_0$ the proof of the upper bound is reduced to prove that

$$\limsup_{\varepsilon\to 0^+}\frac{\log|\log(\mathbb{E}(N_{\omega,k}(E_+ +\varepsilon)-N_{\omega,k}(E_+)))|}{\log\varepsilon}\leqslant-\frac{d}{\nu-d}. \tag{4.14}$$

Lemma 4.1. Let $k\sim\varepsilon^{-\rho}$ with $\rho>1/(\nu-d)$. Define the event,

$$\mathbf{E}_{\varepsilon,\omega}=\left\{\omega;\ \Delta A_{\omega,k}\geqslant-\varepsilon\Delta=-\varepsilon\sum_{i=1}^{d}\partial_{x_i}^2\right\}.$$

Then $\mathbf{E}_{\varepsilon,\omega}$ has a probability at least $1-\mathbb{P}_\varepsilon$ where $\mathbb{P}_\varepsilon$ satisfies

$$\limsup_{\varepsilon\to 0^+}\frac{\log|\log(\mathbb{P}_\varepsilon)|}{\log\varepsilon}\leqslant-\frac{d}{\nu-d}. \tag{4.15}$$

Proof. For $\omega_\gamma^+=\omega^+-\omega_\gamma$ and using the assumption (H.2), we get that there exists $C>0$ such that for any $\varphi\in C_0^\infty(\mathbb{R}^d)$ we have

$$\langle\Delta A_{\omega,k}\varphi,\varphi\rangle\geqslant C\sum_{\gamma\in\mathbb{Z}^d;\,1\leqslant i\leqslant d}\omega_{\tilde\gamma}^+\langle u(\cdot-\gamma)\,\partial_{x_i}\varphi,\partial_{x_i}\varphi\rangle$$

$$\text{where } \tilde\gamma=\gamma \bmod (2k+1)\,\mathbb{Z}^d$$

$$\geqslant C\sum_{\alpha\in\mathbb{Z}^d}\left(\sum_{\gamma\in\mathbb{Z}^d}\omega_{\tilde\gamma}^+ g_-(x-\gamma-\alpha)(1+|\alpha-\gamma|)^{-\nu}\right)|\nabla\varphi|^2$$

$$=\sum_{\alpha\in\mathbb{Z}^d}A_\alpha(\omega)\,g_-(x-\alpha)\,|\nabla\varphi|^2.$$

Here $A_\alpha(\omega)=C\sum_{\gamma\in\mathbb{Z}^d}\omega_{\tilde\gamma}^+(1+|\alpha-\gamma|)^{-\nu}$.

Notice that; $\omega_{\tilde{\gamma}}^{+}$ is $(2k+1)\,\mathbb{Z}^d$ periodic so is $A_\alpha(\omega)$. We set $\mathbb{Z}^d_{2k+1} = \{\alpha \in \mathbb{Z}^d;\ |\alpha| < k\}$. Then we have

$$\mathbb{P}(\{\varDelta A_{\omega,k} \geqslant -\varepsilon \varDelta\}) \geqslant \mathbb{P}(\{\forall \alpha \in \mathbb{Z}^d_{2k+1};\ A_\alpha \geqslant \varepsilon\})$$

$$\geqslant 1 - \sum_{\alpha \in \mathbb{Z}^d_{2k+1}} \mathbb{P}(\{A_\alpha(\omega) \leqslant \varepsilon\}).$$

As the random variables are i.i.d, we have

$$\mathbb{P}(\{\varDelta A_{\omega,k} \geqslant -\varepsilon \varDelta\}) \geqslant 1 - (2k+1)^d\, \mathbb{P}(\{A_0(\omega) \leqslant \varepsilon\}). \tag{4.16}$$

To estimate $\mathbb{P}(\{A_0(\omega) \leqslant \varepsilon\})$ it suffices to follow the same computation done in ref. 11 and based on the Markov's inequality, and the Taylor expansion of e^{-x} to get that

$$\mathbb{P}(\{A_0(\omega) \leqslant \varepsilon\}) \leqslant e^{-\frac{1}{C}\varepsilon^{-\frac{d}{\nu-d}}}. \tag{4.17}$$

By this we complete the proof of Lemma 4.1. ∎

Lemma 4.2. There exists $C > 0$ and $\varepsilon_0 > 0$ (uniform in k and ω) such that, if $0 < \varepsilon < \varepsilon_0$ and ω satisfies $\varDelta A_{\omega,k} \geqslant -C\varepsilon\varDelta$, then for $k \in \mathbb{N}$, one has

$$N_{\omega,k}(E_+) = N_{\omega,k}(E_+ + \varepsilon).$$

Lemma 4.2 says that if $\varDelta A_{\omega,k} \geqslant -C\varepsilon\varDelta$, then the spectrum of $A_{\omega,k}$ does not intersect $(E_+, E_+ + \varepsilon)$ for ε small . This lemma will be proved in the end of this section.

Let us use the last result to finish the proof of the upper bound. Notice that estimate (4.14) is an immediate consequence of Lemmas 4.1 and 4.2. Indeed, picking C as in Lemma 4.2; one computes

$$\begin{aligned}
&\mathbb{E}(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+))\\
&\quad = \mathbb{E}([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega;\ \varDelta A_{\omega,k} \geqslant -C\varepsilon\varDelta\}})\\
&\qquad + \mathbb{E}([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega;\ \varDelta A_{\omega,k} < -C\varepsilon\varDelta\}})\\
&\quad = \mathbb{E}([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega;\ \varDelta A_{\omega,k} < -C\varepsilon\varDelta\}})\\
&\quad \leqslant C_0 \mathbb{P}(\{\omega;\ \varDelta A_{\omega,k} < -C\varepsilon\varDelta\})\\
&\quad = C_0(1 - \mathbb{P}(\mathbf{E}_{C\cdot\varepsilon,\omega})) = C_0 \mathbb{P}_{C\cdot\varepsilon}.
\end{aligned}$$

Here, we have used the fact that $N_{\omega,k}$ is bounded locally uniformly in energy, uniformly in ω, k by C_0. Tanking (4.15) into account we end the proof of (4.14) and consequently the proof of the upper bound. ▮

Proof of Lemma 4.2. Let us take $\Delta A_{\omega,k} > -C\varepsilon\Delta$. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\langle A_{\omega,k}\varphi, \varphi\rangle > E_+$, then taking point (ii) of **(H.0)** into account we get

$$\begin{aligned}\langle A_{\omega,k}\varphi, \varphi\rangle &= \langle A_{\omega^+,k}\varphi, \varphi\rangle + \langle \Delta A_{\omega,k}\varphi, \varphi\rangle \\ &> \langle A_{\omega^+,k}\varphi, \varphi\rangle + C\varepsilon\,|\nabla\varphi|^2 \\ &> \langle A_{\omega^+,k}\varphi, \varphi\rangle + C\varepsilon\varrho_{0,-}E_+.\end{aligned} \tag{4.18}$$

But by assumption **(H.1)**, below the energy E_+ there is a gap in the spectrum of A_{ω^+} of length at least $\delta > 0$; we get that for $\varepsilon < \varepsilon_0 = \frac{\delta}{E_+ C\varrho_{0,-}}$; $A_{\omega,k}$ has no spectrum in $(E_+, E_+ + C\varepsilon E_+\varrho_{0,-})$. So the proof of Lemma 4.2 is ended if we choose $C = \frac{1}{E_+\varrho_{0,-}} > 0$. ▮

4.2. The Lower Bound

The lower bound is proven in the same way as in ref. 16 and consists on proving the following theorem.

Theorem 4.3. Let A_ω be the operator defined by (1). We assume that (H.1) and (H.2) hold. Then, we have

$$\liminf_{\varepsilon\to 0^+} \frac{\log|\log(N(E_+ + \varepsilon) - N(E_+))|}{\log\varepsilon} \geqslant -\frac{d}{\nu - d}. \tag{4.19}$$

Proof. We will sketch the proof, for more details see ref. 16. As by assumption, there is a spectral gap below E_+ of length at least $\delta' > 0$. Thus, for $\varepsilon < \delta'$ we have

$$N(E_+ + \varepsilon) - N(E_+) = N(E_+ + \varepsilon) - N(E_+ - \varepsilon).$$

To prove Theorem 4.3, we will lower bound $N(E_+ + \varepsilon) - N(E_+ - \varepsilon)$. So, for N large, we will show that A_{ω,Λ_N} (A_{ω,Λ_N} is A_ω restricted to Λ_N with Dirichlet boundary conditions) has a large number of eigenvalues in $[E_+ - \varepsilon, E_+ + \varepsilon]$ with a large probability. For this we will construct a family of approximate eigenvectors associated to approximate eigenvalues of A_{ω,Λ_N} in $[E_+ - \varepsilon, E_+ + \varepsilon]$. These functions will be constructed from an eigenvector of A_{ω^+} associated to E_+. Locating this eigenvector in momentum θ, one obtains an approximate eigenfunction of A_{ω,Λ_N}. Notice that the main difference point with the proof given in ref. 16 appears on the choice of the box where we locate this eigenvector in θ. Then we locate this eigenfunction in x in several disjointed places, we get several eigenfunctions two by two orthogonal.

In order to simplify the notations, in what follows we assume that $\theta^0 = 0$ is a point where $E_1(\theta)$ reaches E_+. From refs. 9 and 16 there exists $C > 1$, V a neighborhood of 0 and $f\colon \theta \in V \to f(\cdot, \theta)$ a real analytic function such that, $\|f(\cdot, \theta)\|_{L^2(C_0)} = 1$ and

$$\|(A_{\omega^+}(\theta) - E_+)\, f(\cdot, \theta)\|_{L^2(C_0)} \leqslant C\, |\theta|^2. \tag{4.20}$$

Let $0 < \xi < 1$ be a small constant. Let $\chi \in C_0^\infty(\mathbb{R})$ such that it is positive, supported in $[\frac{\xi}{2}, \xi]$ and $\int_{[\frac{\xi}{2}, \xi]} \chi(t)^2\, dt = 2$.

For $\varepsilon > 0$, we define

$$\mathscr{W}_\varepsilon(\theta) = \varepsilon^{-d/4} \prod_{j=1}^{d} \chi(\varepsilon^{-\frac{1}{2}} \theta_j) \in L^2(\mathbb{T}^*)$$

and

$$\mathscr{W}_\varepsilon^f(\cdot, \theta) = \mathscr{W}_\varepsilon(\theta) \cdot f(\cdot, \theta).$$

Now we estimate

$$\begin{aligned} \|(A_{\omega^+} - E_+)\, \mathscr{W}_\varepsilon^f\|_{\mathscr{H}}^2 &= \frac{1}{\operatorname{vol}(\mathbb{T}^*)} \int_{\mathbb{T}^*} \|(A_{\omega^+}(\theta) - E_+)\, f(\cdot, \theta)\|_{L^2(C_0)}^2\, |\mathscr{W}_\varepsilon(\theta)|^2\, d\theta \\ &\leqslant C^2 \int_{\mathbb{T}^*} |\theta|^4\, |\mathscr{W}_\varepsilon(\theta)|^2\, d\theta \\ &\leqslant C^2 \varepsilon^2 \int_{[\frac{\xi}{2}, \xi]^d} |\theta|^4 \prod_{j=1}^{d} \chi^2(\theta_j)\, d\theta \\ &\leqslant \frac{\varepsilon^2}{8}, \qquad \text{if } \xi \text{ is small enough.} \end{aligned} \tag{4.21}$$

For $\beta \in \mathbb{Z}^d$, we define

$$\mathscr{W}_{\varepsilon, \beta}^f(\cdot, \theta) = e^{-i\beta \cdot \theta} \mathscr{W}_\varepsilon^f(\cdot, \theta) \quad \text{and} \quad \mathscr{W}_{\alpha, \varepsilon, \beta, \zeta}^f(\cdot, \theta) = e^{-i\beta \cdot \theta} (\Pi_{\Lambda_\alpha(\zeta)} \mathscr{W}_\varepsilon^f)(\cdot, \theta),$$

where $\Lambda_\alpha(\zeta)$ is the cube defined by

$$\Lambda_\alpha(\zeta) = \{\gamma \in \mathbb{Z}^d;\ \text{for } 1 \leqslant j \leqslant d;\ |\gamma_j| \leqslant \zeta^{-(\frac{1}{\nu - d} + \alpha)}\},$$

and $\Pi_{\Lambda_\alpha(\zeta)}$ is the orthogonal projection on $\Lambda_\alpha(\zeta)$.

We set

$$\mathcal{U}^f_{\varepsilon,\beta}(x) = \int_{\mathbb{T}^*} \mathcal{W}^f_{\varepsilon,\beta}(x,\theta)\, d\theta \qquad \text{and} \qquad \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x) = \int_{\mathbb{T}^*} \mathcal{W}^f_{\alpha,\varepsilon,\beta,\zeta}(x,\theta)\, d\theta.$$

For N large and well chosen β and $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$, $\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}$ will be an approximate eigenfunction of A_{ω,Λ_N} associated with an approximate eigenvalue in the interval $[E_+ - \varepsilon, E_+ + \varepsilon]$.

We show initially that $\|\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\|_{L^2(\mathbb{R}^d)} > C > 0$. Note that

$$\begin{aligned}(\mathrm{vol}(\mathbb{T}^*))\, \|\mathcal{U}^f_{\varepsilon,\beta}\|^2_{L^2(\mathbb{R}^d)} &= \|\mathcal{W}^f_{\varepsilon,\beta}\|^2_{\mathcal{H}} \\ &= \int_{\mathbb{T}^*} \|f(\cdot,\theta)\|^2_{L^2(C_0)}\, |\mathcal{W}_\varepsilon(\theta)|^2\, d\theta \geqslant 2^d.\end{aligned}$$

As in ref. 9 using the non-stationary phase we see that $\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}$ and $\mathcal{U}^f_{\varepsilon,\beta}$ are close to each others. More precisely, for any $n\in\mathbb{N}$ and $\beta\in\mathbb{Z}^d$, there exists $C_n > 0$ such that

$$(\mathrm{vol}(\mathbb{T}^*))\cdot \|\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta} - \mathcal{U}^f_{\varepsilon,\beta}\|_{L^2(\mathbb{R})} = \|\mathcal{W}^f_{\alpha,\varepsilon,\beta,\zeta} - \mathcal{W}^f_{\varepsilon,\beta}\|_{\mathcal{H}} \leqslant C_n \varepsilon^{-n/2} \zeta^{n(\frac{1}{\nu-d}+\alpha)}. \tag{4.22}$$

So, for $\zeta = \varepsilon$ small enough, we get

$$\|\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\|_{L^2(\mathbb{R}^d)} \geqslant 1.$$

Now we have to look to the conditions for which we have

$$\|(A_\omega - E_+)\, \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\|^2 \leqslant \varepsilon^2.$$

Note that

$$\begin{aligned}\|(A_{\omega,\Lambda_N} - E_+)\, \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\|^2 &\leqslant \|(A_\omega - E_+)\, \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\|^2 \\ &\leqslant 2\, \|(A_{\omega^+} - E_+)\, \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\|^2 + 2\, \|\Delta A_\omega \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\|^2.\end{aligned} \tag{4.23}$$

Equations (4.21) and (4.22) give the bound on the first member of (4.23), it just remains to us to control the second term. For $\omega^+_\gamma = \omega^+ - \omega_\gamma$, we have

$$\begin{aligned}\|(\Delta A_\omega)\, \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\|^2 &\leqslant 2\, \Bigg\| \sum_{\gamma\in\mathbb{Z}^d,\, 1\leqslant i\leqslant d} \omega^+_\gamma (\partial_{x_i} u)(\cdot - \gamma)\cdot(\partial_{x_i} \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}) \Bigg\|^2 \\ &\quad + 2\, \Bigg\| \sum_{\gamma\in\mathbb{Z}^d,\, 1\leqslant i\leqslant d} \omega^+_\gamma u(\cdot - \gamma)\, \partial^2_{x_i} \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta} \Bigg\|^2.\end{aligned} \tag{4.24}$$

To estimate (4.24), one needs the following lemmas, proven in ref. 16.

Lemma 4.4 [16]. There exists $K > 0$, such that

$$\left\| \sum_{\gamma \in \mathbb{Z}^d,\, 1 \leqslant i \leqslant d} \omega_\gamma^+ (\partial_{x_i} u)(\cdot - \gamma) \cdot (\partial_{x_i} \mathscr{U}^f_{\alpha, \varepsilon, \beta, \zeta}) \right\|^2 \leqslant \varepsilon^4 + K(\varepsilon^{\alpha(\nu - d)} \cdot \varepsilon + \sup_{\gamma \in \beta + 2\Lambda_\alpha(\varepsilon)} \omega_\gamma^+)^2. \tag{4.25}$$

Lemma 4.5 [16]. There exists $K > 0$, such that

$$\left\| \sum_{\gamma \in \mathbb{Z}^d,\, 1 \leqslant i \leqslant d} \omega_\gamma^+ u(\cdot - \gamma)\, \partial^2_{x_i} \mathscr{U}^f_{\alpha, \varepsilon, \beta, \zeta} \right\|^2 \leqslant \varepsilon^4 + K(\varepsilon^{\alpha(\nu - d)} \cdot \varepsilon + \sup_{\gamma \in \beta + 2\Lambda_\alpha(\varepsilon)} \omega_\gamma^+)^2. \tag{4.26}$$

Now, combining (4.25), (4.26) and taking (4.24) into account we get that there exists $K > 0$ such that

$$\|(\Delta A_\omega)\, \mathscr{U}^f_{\alpha, \varepsilon, \beta, \varepsilon}\|^2 \leqslant \varepsilon^3 + K(\varepsilon^{\alpha(\nu - d)} \cdot \varepsilon + \sup_{\gamma \in \beta + 2\Lambda_\alpha(\varepsilon)} \omega_\gamma^+)^2. \tag{4.27}$$

By (4.21),(4.22), and (4.27), it follows that:

$$\|(A_\omega - E_+)\, \mathscr{U}^f_{\alpha, \varepsilon, \beta, \varepsilon}\|^2 \leqslant \frac{\varepsilon^2}{2} + K(\varepsilon^{\alpha(\nu - d)} \cdot \varepsilon + \sup_{\gamma \in \beta + 2\Lambda_\alpha(\varepsilon)} \omega_\gamma^+)^2. \tag{4.28}$$

Now, for N large, we may divide Λ_N into $N(\varepsilon)$ disjoints cubes of size $2\,\Lambda_\alpha(\varepsilon)$. One has

$$N(\varepsilon) \simeq \frac{(2N)^d}{\varepsilon^{-d(\frac{1}{\nu - d} + \alpha)}}; \tag{4.29}$$

and

$$\bigcup_{j=1}^{N(\varepsilon)} (\beta_j + \Lambda_\alpha(\varepsilon)) \subset \Lambda_N \quad \text{and for } j \neq j',\ (\beta_j + 2\,\Lambda_\alpha(\varepsilon)) \cap (\beta_{j'} + 2\,\Lambda_\alpha(\varepsilon)) = \emptyset.$$

This implies that for $j \neq j'$, $\mathscr{U}^f_{\alpha, \varepsilon, \beta_j, \varepsilon}$ and $\mathscr{U}^f_{\alpha, \varepsilon, \beta_{j'}, \varepsilon}$ are orthogonal.

We denote the counting function of the eigenvalues of A_{ω, Λ_N} below E by $\Theta_{\Lambda_N}(E)$, then

$$\begin{aligned}
&\mathbb{E}(\Theta_{\Lambda_N}(E + \varepsilon) - \Theta_{\Lambda_N}(E - \varepsilon)) \\
&\quad = \mathbb{E}(\#\{\text{eigenvalues of } \Pi_N A_\omega \Pi_N \text{ in } [E_+ - \varepsilon, E_+ + \varepsilon]\}) \\
&\quad \geqslant \mathbb{E}(\#\{1 \leqslant j \leqslant N(\varepsilon);\ \|(A_\omega - E_+)\, \mathscr{U}^f_{\alpha, \varepsilon, \beta_j, \varepsilon}\|_{L^2(\mathbb{R}^d)} \leqslant \varepsilon\}) \\
&\quad \geqslant \mathbb{E}\left(\sum_{j=1}^{N(\varepsilon)} B_j(\omega)\right),
\end{aligned} \tag{4.30}$$

where

$$B_j(\omega)=\begin{cases}1 & \text{if}\quad K(\varepsilon^{\alpha(\nu-d)}\cdot\varepsilon+\sup\limits_{\gamma\in\beta_j+2\Lambda_\alpha(\varepsilon)}\omega_\gamma^+)^2\leqslant\dfrac{\varepsilon^2}{2},\\ 0 & \text{if not.}\end{cases}$$

The $(B_j)_{1\leqslant j\leqslant N(\varepsilon)}$ are independent, identically distributed, Bernoulli random variables. So Eqs. (4.29) and (4.30), imply that there exists $C>0$, such that one has

$$\begin{aligned}&N_{\Lambda_N}(E+\varepsilon)-N_{\Lambda_N}(E-\varepsilon)\\&\quad=\frac{1}{((2N+1))^d}\,\mathbb{E}(\#\{\text{eigenvalues of }\Pi_N A_\omega\Pi_N\text{ in }[E_+-\varepsilon,E_++\varepsilon]\})\\&\quad\geqslant\frac{N(\varepsilon)}{(2N+1)^d}\,\mathbb{P}(B_1=1)\geqslant\frac{1}{C}\,\varepsilon^{d(\frac{1}{\nu-d}+\alpha)}\mathbb{P}(B_1=1).\end{aligned}$$

Hence, taking the limit $N\to\infty$, we get that, for $\varepsilon>0$ small, we obtain

$$N(E_++\varepsilon)-N(E_+-\varepsilon)\geqslant\frac{1}{C}\,\varepsilon^{d(\frac{1}{\nu-d}+\alpha)}\mathbb{P}(B_1=1).\tag{4.31}$$

It just remains to estimate $\mathbb{P}(B_1=1)$. If for $1\leqslant j\leqslant N(\varepsilon)$, and any $\gamma\in\beta_j+2\,\Lambda_\alpha(\varepsilon)$; one has $\omega_\gamma^+\leqslant\frac{\varepsilon}{2K}$, then for ε rather small

$$K(\varepsilon^{\alpha(\nu-d)}\cdot\varepsilon+\sup_{\gamma\in\beta_j+2\Lambda_\alpha(\varepsilon)}\omega_\gamma^+)^2\leqslant\varepsilon^2\cdot K\left(\varepsilon^{\alpha(\nu-d)}+\frac{1}{2K}\right)^2\leqslant\frac{\varepsilon^2}{2}.$$

As the random variables are independent identically distributed, one has the estimate

$$\mathbb{P}(B_j=1)\geqslant\left(\mathbb{P}\left(\omega_0^+\leqslant\frac{\varepsilon}{2K}\right)\right)^{2\#\Lambda_\alpha(\varepsilon)}.$$

So, taking the double logarithm of (4.31), using assumption (H.3) and the fact that $\#\Lambda_\alpha(\varepsilon)=\varepsilon^{-d(\frac{1}{\nu-d}+\alpha)}$, we get that

$$\lim_{\varepsilon\to0^+}\frac{\log|\log(N(E_++\varepsilon)-N(E_+))|}{\log\varepsilon}\geqslant-\frac{d}{\nu-d}-d\alpha.\tag{4.32}$$

The Eq. (4.32) is true for any $\alpha>0$, by taking α small we end the proof of Theorem 4.3. ∎

ACKNOWLEDGMENTS

The author would like to thank professor Alain GRIGIS for his hospitality at the university of Paris 13 where a part of this work has been done.

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